Rope Pull and Conservation of Energy — resolving an apparent paradox in the solution to a lower-division physics problem

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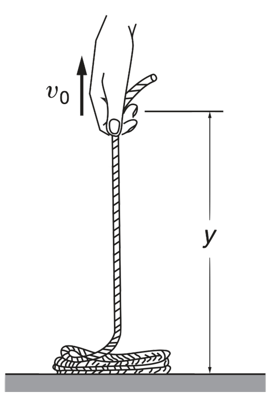
One problem from Kleppner/Kolenkow's *Introduction to Mechanics* textbook has a solution that seems to violate conservation of energy. Solutions found online don't offer much explanation — most state that "inelastic processes don't conserve energy" or similar. This explanation is unsatisfying, as there is no apparent physical mechanism in the problem that could account for the energy dissipation. This paper explores this problem in greater detail, using a 2D and 3D model for the rope. We find two relevant, non-negligible (not obvious) results: 1. that the table can provide an impulse to the rope without giving it any mechanical energy, and 2. that the rope can pick up a non-negligible rotational kinetic energy.

I. INTRODUCTION

The problem of interest is problem 5.19 from Kleppner and Kolenkow's *Introduction to Mechanics*, 2nd Edition [1]. (In the first edition, it is problem 4.19 — the wording is identical except for trivial modifications.) This problem appears at the end of a chapter on work/energy/power, which follows the chapter on momentum. It is one of the more difficult problems seen by students in a lower-division mechanics class:

5.19 *Coil of rope*
A uniform rope of mass density λ per unit length is coiled on a smooth horizontal table. One end is pulled straight up with constant speed v_0, as shown.



(a) Find the force exerted on the end of the rope as a function of height y.
(b) Compare the power delivered to the rope with the rate of change of the rope's total mechanical energy.

The solution to part (a) has two terms in the answer:

$$F_{\text{app}} = F_1 + F_2 = \lambda y g + \lambda v_0^2 \tag{1}$$

F_{app} is the magnitude of this upwards force. The force F_1 is that required to support the weight of rope already off the ground. This section of rope is traveling at constant velocity v_0 upwards, and has no net force on it. The force F_2 is that required to accelerate a length of rope Δx (of mass $\Delta m = \lambda \Delta x$) from rest to a final velocity v_0 upwards, in a time Δt, where $v_0 = \Delta x/\Delta t$. Although a student could produce F_2 by assuming a constant average acceleration and using Newton's 2nd law (with $F_{\text{net}} = (\Delta m)a$), it is more straightforward to use the impulse-momentum theorem. Note the force required to support the weight of Δm is infinitesimal in the $\Delta t \to 0$ limit, and is not included in either F_1 or F_2.

The solution to part (b) has two parts: first, the power delivered to the rope directly from the applied force,

$$P_{\text{app}} = \vec{F}_{\text{app}} \cdot \vec{v} = F_{\text{app}} v_0 = \lambda y g v_0 + \lambda v_0^3, \tag{2}$$

and second, the time rate of change of the rope's total mechanical energy: dE_{mech}/dt. To find the latter, we first write $E_{\text{mech}}(t)$ with time-dependence made explicit. Note $y(t) = v_0 t$, and let $m(t) = \lambda y(t)$ be the mass of rope above the ground at time t:

$$\begin{aligned} E_{\text{mech}} &= U_{\text{g, CM}} + K \\ &= m(t) g \left(\frac{1}{2} y(t)\right) + \frac{1}{2} m(t) v_0^2 \\ &= \frac{1}{2}\lambda v_0^2 g t^2 + \frac{1}{2}\lambda v_0^3 t \end{aligned} \tag{3}$$

$$\begin{aligned} \frac{dE_{\text{mech}}}{dt} &= \lambda v_0^2 g t + \frac{1}{2}\lambda v_0^3 \\ &= \lambda y g v_0 + \frac{1}{2}\lambda v_0^3. \end{aligned} \tag{4}$$

The interesting result is that the power put into the system by the applied force, P_{app}, is greater than the time rate of change of the mechanical energy of the system. While non-conservation of mechanical energy is not unheard of in introductory mechanics problems (e.g., all inelastic collisions), the system here is relatively simple, and usually for problems of this sort the mechanism by which energy could be lost is clear. For example, the problem right before this one (5.18 *Sand and Conveyor Belt* in the 2nd edition) has a similar discrepancy, but the difference can be accounted for with the standard model of friction covered in introductory classes. For our coil of rope problem, however, it is not apparent that friction can explain what is going on. It is the purpose of this letter to study where this "missing" energy goes.

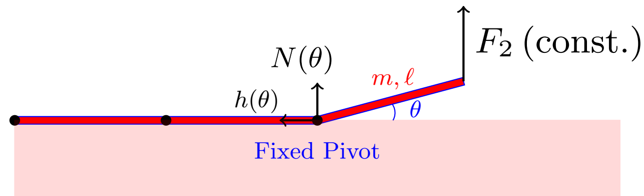
In studying the answer to this question using different models of the problem, we uncover some unexpected results. For example, under a straightforward 2D model of chain links rotating to a vertical position upon "launch" of Δm, we learn that the force F_2 can be significantly different from either λv_0^3 (that seen in Eq. (2)) or half of this value (that required to be consistent with Eq. (4)). The exact value of F_2 is highly sensitive to the parameters of the model, but we learn that the table/ground plays an important role in the solution, since it is able to provide an impulse to the rope without providing any energy. Afterwards, we'll turn our attention to a 3D model of the coil unwinding on the table. In Sec. III we find there is a non-negligible "rotational energy," even in the limit that the radius of the coil goes to zero. Incorporating the fact that the impulse from the ground can provide some force without adding to the mechanical energy of the system, we'll see that, under some simplifying assumptions, the actual value of F_2 is $\frac{2}{3}\lambda v_0^3$.

The discrepancy is related to the term F_2 and not F_1. This term contains no reference to the acceleration due to gravity, g, and so we can eliminate some possible explanations for the discrepancy. This also helps us simplify the analysis — we ignore gravitational force in the following section, Sec. II. Also, note that the term $F_2 = \lambda v_0^2$ is constant (as is its contribution to the power provided); we will continue taking F_2 constant in this paper.

II. 2D MODEL: CHAIN OF RODS

The first model we will study is a rope consisting of several tiny, rigid chain links ("rods"). The rod being lifted is subject to a constant vertical force F_2 at one end and pivoted at the other end. The constraint that the rod not go into the table requires a normal force which is proportional to F_2 and (at least initially) upwards. This realization already tells us that we will have to modify our expression for F_2: the normal force will provide an upwards impulse to the rod, but will not provide any mechanical energy to the rod. This is in contrast to the solution in Sec. I, whereby F_2 was entirely responsible for the net impulse and the net energy delivered to the rope. It seems likely, then, that F_2 is actually not as big as previously thought, and Eq. (2) will be modified.

Ideally we would just include the single force F_2 in the model. However, pulling upwards on one end of the rod (to the right of the point of percussion, $\ell/3$ from the right end of the rod) would normally cause



the left end of the rod to initially move downwards. There is therefore a normal force $N(\theta)$ from the table, initially pointing upwards to enforce the constraint that the left end of the rod not move down into the table. In addition, we assume there is a leftwards horizontal force of constraint $h(\theta)$ keeping the pivot fixed — this horizontal constraint force could from from friction (at least initially[1]) or from the next chain link.

In this section we'll use m for this chain link / rod (rather than the Δm introduced in the last section), and we'll use ℓ for the length of the rod (rather than Δx or Δy). The rotational analog of Newton's 2nd law (with the left end of the rod taken as the pivot), the impulse-momentum theorem (or, equivalently, Newton's 2nd law), conservation of energy, and the constraint that the left end of the rod remains fixed give the following four equations:

$$\boxed{\vec{\tau}} \qquad F_2 \ell \cos\theta = I\alpha = \left(\frac{1}{3} m \ell^2\right)\ddot{\theta} \tag{5}$$

$$\boxed{\vec{p}} \qquad \frac{dv_{\mathrm{CM},\,y}}{dt} = \frac{1}{m}\Big(F_2 + N(\theta)\Big) \tag{6}$$

$$\boxed{E} \qquad F_2 \ell \sin\theta = \frac{1}{2} I \omega^2 = \frac{1}{6} m \ell^2 \omega^2 \tag{7}$$

$$\boxed{\text{constraint}} \qquad v_{\mathrm{CM}} = \omega \frac{\ell}{2} \tag{8}$$

The $+y$-direction is taken to be vertically upwards for Eq. (6). Eq. (5) can also be found by taking the time-derivative of Eq. (7), so this is a set of three independent equations. The dependence of the normal force on θ is explicitly shown in order to contrast with the constant F_2. Of course, $\omega = \dot{\theta}$ and $\ddot{\theta}$ both depend on θ (or time) as well.

We could like to solve for $N(\theta)$. Take Eq. (8) (with $v_{\mathrm{CM},\,y} = v_{\mathrm{CM}} \cos\theta$) and find the time derivative to obtain

$$\frac{v_{\mathrm{CM},\,y}}{dt} = \frac{\ell}{2}\Big[\ddot{\theta}\cos\theta - \omega^2 \sin\theta\Big] \tag{9}$$

If we use Eq. (6) to substitute for the left-hand side, and

[1] This statement comes from the fact that $N(\theta)$ decreases as angle increases, while $h(\theta)$ increases as angle increases (both approximately linearly for small angles). When $h(\theta)/N(\theta) > \mu_s$, static friction alone cannot be the sole source of $h(\theta)$.

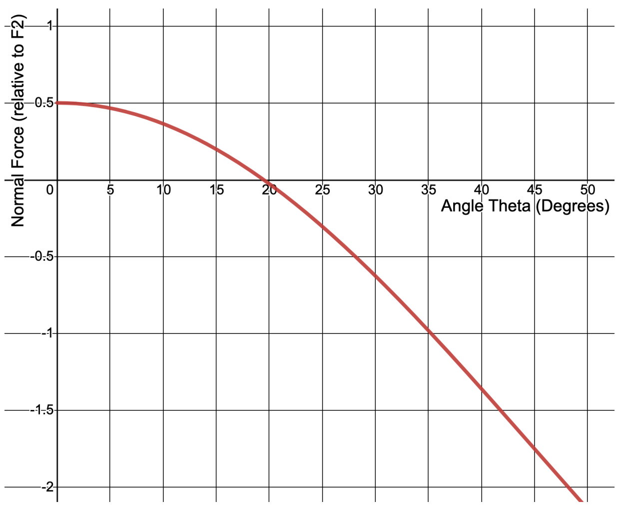
Eq. (5) to substitute for $\ddot{\theta}$ in the right-hand side, we get

$$\frac{1}{m}\Big[F_2 + N(\theta)\Big] = \frac{\ell}{2}\left[\frac{3F_2}{m\ell}\cos^2\theta - \omega^2\sin\theta\right] \tag{10}$$

Plugging in $\omega^2 = 6F_2\sin\theta/(m\ell)$ from Eq. (7), we can solve for $N(\theta)$ to obtain

$$N(\theta) = F_2\left[\frac{1}{2} - \frac{9}{2}\sin^2\theta\right] \tag{11}$$

The normal force required to enforce the constraint decreases from $N(0) = \frac{1}{2}F_2$ at $\theta = 0°$ to $N(\theta^c) = 0$ at $\theta = \theta^c \equiv \sin^{-1}(1/3) \approx 19.5°$ (and becomes negative for larger angles):



The rod lifts off the ground early enough that the normal force does not seem to play a major role in the total impulse delivered to the rod. Quantitatively, we can calculate this (upwards) impulse delivered to the rod from the normal force during the interval for which $N(\theta) \geq 0$, and compare it to the (upwards) impulse from F_2 during the same interval:

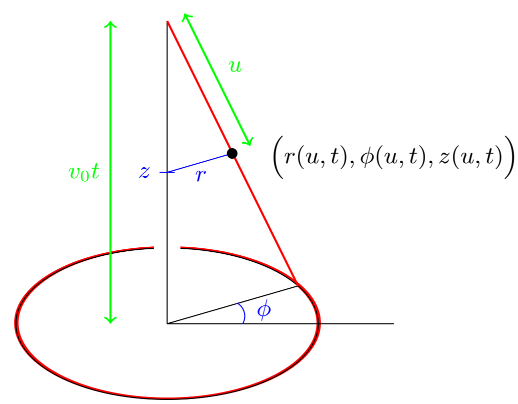
$$\begin{aligned} J_N &= \int N dt = \int_0^{\theta^c} \frac{N(\theta)}{\omega}\, d\theta \approx 0.190\sqrt{F_2\ell m} \\ J_{F_2} &= \int F_2\, dt = F_2\int_0^{\theta^c} \frac{d\theta}{\omega} \approx 0.477\sqrt{F_2\ell m} \end{aligned} \tag{12}$$

To resolve the discrepancy in the original problem, these impulses need to be equal. Not only is this not true here, but this is compounded by two facts: 1. the applied force F_2 continues past this angle, as the link is still accelerating upwards, and 2. the normal force could become effectively negative for angles above θ^c, since the neighboring link could pull downwards on the rod (and, in turn, the upwards normal force on the neighboring link would decrease by the same amount, assuming that neighboring link stays on the table).

All this is to say that we will need a more complicated model to better understand what is going on in terms of impulse and energy. It is unreasonable to expect that the pivot will stay fixed for angles much larger than θ^c. It appears likely that, in this model, the extra energy will go towards internal motion of the rods (i.e., different modes of oscillation, moving transverse to the upwards motion of the center-of-mass). If a suitable theoretical model is untenable, we could use a simulation to gain some traction, at least numerically.

III. 3D MODEL: COIL OF ROPE

We now turn our attention towards a model that more directly corresponds to the problem from the textbook. We'll take the coil of rope to have radius R (which we take to zero at the end of the computation), and we will use cylindrical coordinates (r, ϕ, z) with the rope lying in the xy-plane concentric with the origin (i.e., we switch from "y" to "z"). We approximate the portion of rope that has been lifted to be a straight line:



We will parameterize the rope with u, the length along the rope from the top (where the applied force is pulling the rope upwards). With this parameterization, $z(u,t)$, $r(u,t)$, and $\phi(u,t)$ are as follows:

$$\frac{v_0 t - z}{u} = \frac{v_0 t}{\sqrt{v_0^2 t^2 + R^2}} \implies z(u,t) = v_0 t - \frac{v_0 t u}{\sqrt{v_0^2 t^2 + R^2}} \tag{13}$$

Keep in mind that, even though the top of the rope is at z-coordinate $v_0 t$, the $z(u,t)$ above is describing any point on the straight-line segment of rope (a distance u from the top). Like $z(u,t)$, $r(u,t)$ can be found by exploiting similar triangles:

$$\frac{r}{u} = \frac{R}{\sqrt{v_0^2 t^2 + R^2}} \implies r(u,t) = \frac{Ru}{\sqrt{v_0^2 t^2 + R^2}} \tag{14}$$

We can find $\phi(u,t)$ by utilizing the fact that the arclength subtended so far on the table is equal to the total length of rope lifted off the table[2]:

$$\phi R = \sqrt{v_0^2 t^2 + R^2} - R \implies \phi(u,t) = \sqrt{1 + \frac{v_0^2 t^2}{R^2}} - 1 \tag{15}$$

A. Total Mechanical Energy

In this subsection we find the total gravitational potential energy ($U_g(t)$) and the total kinetic energy ($K(t)$) of the rope as a function of time. We will look at forces in the next subsection.

First, we'll look at the total gravitational potential energy at a given time t by integrating over the segment of rope above the table. Note $dm = \lambda du$ and the length of rope above the ground (the maximum value of u) is $L(t) = \sqrt{v_0^2 t^2 + R^2}$:

$$\begin{aligned} U_g(t) &= \int_{\text{rope}} gz\, dm \\ &= \int_0^{L(t)} g v_0 t \left(1 - \frac{u}{L(t)}\right) \lambda\, du \\ &= g v_0 t \lambda \frac{L(t)}{2} = \frac{1}{2} g v_0 t \lambda \sqrt{v_0^2 t^2 + R^2} \end{aligned} \tag{16}$$

$$\lim_{R\to 0} U_g(t) = \frac{1}{2} g v_0^2 t^2 \lambda \checkmark$$

This is the answer that we expected from our original solution. This is not too surprising, as the discrepancy is in F_2 and not F_1.

Next, we'll look at the total kinetic energy at a given time t via a similar integration:

$$\begin{aligned} K(t) &= \int_{\text{rope}} \frac{1}{2}\left(\vec{v}\cdot\vec{v}\right) dm \\ &= \int_0^{L(t)} \frac{1}{2}\left(\dot{z}^2 + \dot{r}^2 + r^2\dot{\phi}^2\right)\lambda\, du \end{aligned} \tag{17}$$

We'll look at these pieces term-by-term, where $K(t) = K_z(t) + K_r(t) + K_\phi(t)$ are the three terms that appear in Eq. (17), above. We'll need the following time-derivatives of the variables:

$$\begin{aligned} \dot{z} &= v_0 - \frac{R^2}{L^3} v_0 u \\ \dot{r} &= -\frac{Ru}{L^3} v_0^2 t \\ \dot{\phi} &= \frac{v_0^2 t}{RL} \end{aligned} \tag{18}$$

t is a constant parameter in the integral over u; the dot over the coordinate really signifies a partial derivative with repsect to time holding u constant (which will then be used in the integral over u).

Kinetic Energy term 1, $K_z(t)$:

$$\begin{aligned} K_z(t) &= \frac{\lambda}{2}\int_0^L \left(v_0 - \frac{R^2}{L^3} v_0 u\right)^2 du \\ &= \frac{1}{2}\lambda v_0^2 L \left(1 - \frac{R^2}{L^2} + \frac{1}{3}\frac{R^4}{L^4}\right) \end{aligned} \tag{19}$$

$$\lim_{R\to 0} K_z(t) = \frac{1}{2}\lambda v_0^3 t \checkmark$$

Note that the limit of $R/L = R/\sqrt{v_0^2 t^2 + R^2}$ is zero in the $R \to 0$ limit. The checkmark signifies that this is our expected result for the total kinetic energy of the rope, as we saw in Eq. (3).

Kinetic Energy term 2, $K_r(t)$:

$$\begin{aligned} K_r(t) &= \frac{\lambda}{2}\int_0^L \dot{r}^2\, du \\ &= \frac{\lambda}{6}\frac{R^2 v_0^4 t^2}{L^3} \end{aligned} \tag{20}$$

$$\lim_{R\to 0} K_r(t) = 0 \checkmark$$

The checkmark signifies that there is no additional kinetic energy term here (i.e., no corrections to our original answer).

Kinetic Energy term 3, $K_\phi(t)$:

$$\begin{aligned} K_\phi(t) &= \frac{\lambda}{2}\int_0^L \frac{v_0^4 t^2}{L^4} u^2\, du \\ &= \frac{\lambda}{6}\frac{v_0^4 t^2}{L} \end{aligned} \tag{21}$$

$$\lim_{R\to 0} K_\phi(t) = \frac{1}{6}\lambda v_0^3 t \neq 0 \text{ (!?)}$$

This is nonzero, which is surprising. Even in the $R \to 0$ limit, there is some nonzero kinetic energy associated with the rotational motion. Of course, this is a new

[2] Whether or not we subtract out the R in this total length of rope is not so important, as our discussion really only depends on ϕ.

result, not seen in the 2D model.

The total kinetic energy in the $R \to 0$ limit is then

$$\lim_{R\to 0} K(t) = \frac{2}{3}\lambda v_0^3 t \text{ (!)} \tag{22}$$

B. Total Force

To find the net force on the rope (that is, the portion of the rope that is in the air), we first find an expression for $\vec{p}_{\rm tot}$ and then differentiate with respect to time:

$$\vec{p}_{\rm tot} = \int_{\rm rope} \vec{v}\, dm = \lambda \int_0^{L(t)} \vec{v}\, du \tag{23}$$

where $\vec{v} = \dot{z}\hat{z} + r\dot{\phi}\hat{\phi} + \dot{r}\hat{r}$. These three components evaluate to

$$\begin{aligned} p_{{\rm tot},\,z} &= v_0\lambda L\left(1 - \frac{R^2}{2L^2}\right) \\ p_{{\rm tot},\,\phi} &= \frac{1}{2}\lambda v_0^2 t \\ p_{{\rm tot},\,r} &= -\frac{1}{2}\lambda\left(\frac{R}{L}\right) v_0^2 t \end{aligned} \tag{24}$$

Taking a derivative with respect to time, and taking the $R \to 0$ limit, we find that the net force on the rope as a function of time is

$$\vec{F}_{\rm net\,on\,rope} = \frac{d\vec{p}_{\rm tot}}{dt} = \left(\lambda v_0^2\right)\hat{z} + \left(\frac{1}{2}\lambda v_0^2\right)\hat{\phi} \tag{25}$$

This net force is the sum of three contributions: a constant upwards applied force at the top of the rope (which we have been calling F_2 in magnitude), the force from the ground/table (and possibly the other parts of the rope), and the gravitational force:

$$\begin{aligned} \vec{F}_{\rm net\,on\,rope} &= \vec{F}_{\rm app} + \vec{F}_{\rm ground} + \vec{F}_g \\ F_{{\rm net},\,z} &= F_{{\rm app},\,z} + F_{{\rm ground},\,z} - \lambda z g \end{aligned} \tag{26}$$

If the applied force satisfies $F_{{\rm app},\,z} = \frac{2}{3}\lambda v_0^2 + \lambda z g$, and if ground/table has normal component $F_{{\rm ground},\,z} = \frac{1}{3}\lambda v_0^2$, then there is no longer any discrepancy. Interestingly, this normal force of magnitude half the applied force (ignoring the gravitational piece) is the same situation we encountered in Sec. II, so long as the rod was horizontal. Therefore, there is no discrepancy so long as it is a good approximation that the effective fulcrum for the piece of rope rising off the table is far enough away that the section lifting off the ground is very nearly horizontal. In that case, we have an applied power of

$$P_{\rm app} = \frac{dE_{\rm mech}}{dt} = \lambda z g v_0 + \frac{2}{3}\lambda v_0^3 \tag{27}$$

The force from the ground/table makes up the remaining impulse delivered to the rope. This force has both z and ϕ components.

If, however, the ground/table is unable to provide this impulse, then the applied force must make up the difference, and there will be a greater applied power than the time rate-of-change of the rope's mechanical energy. Presumably, the "loss" of energy goes into internal energy of the rope (swinging side-to-side in addition to moving upwards).

IV. CONCLUSIONS

We explored a problem/solution from Kleppner/Kolenkow's Introduction to Mechanics textbook (5.19 *Coil of rope*). While the standard solution has an apparent paradox (or, an implied loss of mechanical energy to dissipative forces), where the applied power is greater than the rate-of-change of the system's mechanical energy, we give two other explanations for what happens:

1. Because the table can provide an impulse to the rope without supplying any energy, the applied power is not as great, and there is a reduced discrepancy between applied power and the rate at which mechanical energy is supplied to the rope.

2. Because the rope "unwinds" as it is lifted, it can have some rotational motion — the azimuthal component of this motion has a non-negligible kinetic energy associated with it.

(Combining the two above effects:) As compared to the usual answer, the applied force is smaller and the system's mechanical energy is larger, and there is not necessarily any energy dissipation.

[1] D Kleppner and R Kolenkow. An introduction to mechanics. 2013.